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ON THE MAXIMUM AND ABSORPTION TIME OF LEFT-CONTINUOUS RANDOM WALK

by

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ABSTRACT

In a recent paper P. J. Green obtained some conditional limit theorems for the absorption time of left-continuous random walk. His methods required certain distributions to have exponentially decreasing tails. Here we take a different approach which will produce Green's results under minimal conditions. Limit theorems are given for the maximum as the initial position of the random walk tends to infinity.

Key words: left continuous random walk; maximum; absorption time; limit theorems; local limit theorems; renewal theorems.

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1. INTRODUCTION

We consider a left-continuous random walk $\{S_n : n=0,1,\ldots\}$ on the non-negative integers with $\{0\}$ as an absorbing state. Specifically, for $j = 0,1,\ldots,$ let

$$P\{S_{n+1} = S_n - 1 + j \mid S_n\} = p_j \quad (S_n > 0), \; = \delta_{j1} \quad (S_n = 0)$$

where we assume $p_0 > 0$, $p_0 + p_1 < 1$ and that $\{p_j\}$ has unit maximal span. Write $f(t) = \sum_{j=0}^{\infty} p_j t^j$. Recently Green (1976) has obtained expressions for the distribution of (M,N) where N is the time to absorption and $M = \max\{S_n ; n \leq N\}$. In the cases $\alpha = f'(1-) \leq 1$ he obtained some results on the tail behavior of M and limit theorems for N as $M \to \infty$. However his methods of proof require conditions which appear not to be necessary for the validity of the results. For example when $\alpha = 1$ his stated results involve only $f''(1-)$ which must necessarily be finite, but their proofs require that $f(s)$ exists for $|s| < 1 + \varepsilon$ for some $\varepsilon > 0$.

We shall derive alternative representations for some of Green's quantities and use these together with renewal theorems and local limit theorems for the n-step transition probabilities of $\{S_n\}$ when $\alpha = 1$ and, when $\alpha < 1$, for an associated random walk with positive drift, to obtain Green's results under minimal conditions. When $\alpha < 1$ we also obtain limit theorems for N (as $M \to \infty$) under some weaker moment assumptions (Theorems 4 and 5) and when $\alpha \leq 1$ we obtain limit theorems for M as $S_0 \to \infty$.

2. THE CASE OF ZERO DRIFT

For $0 \leq s \leq 1$; $k,m = 0,1,\ldots,$ let

$$\Pi_k(m,s) = E(s^N; M \leq m | S_0 = k). \tag{1}$$

Green (1976) has shown that

$$\Pi_k(m,s) = U_{m-k}(s)/U_m(s)$$

where

$$\sum_{j=0}^{\infty} U_j(s)t^j = \frac{p_0 s}{sf(t)-t}$$

which exists if $|t| < g(s)$ where $g(s)$ is the least non-negative solution of $sf(t) = t$. As is well known, $(g(s))^k$ is the probability generating function (p.g.f.) of N when $S_0 = k$. Now define $\bar{f}(t) = sf(tg(s))/g(s)$ $(0 \leq t \leq 1)$ which is a p.g.f. for each $s \in (0,1]$. Define the sequence of functions $\{u_j(s): j=0,1,\ldots\}$ by $u_0(s) \equiv 1$ and

$$\sum_{j=1}^{\infty} u_j(s)t^j = \frac{p_0 s}{g(s)(\bar{f}(t)-t)} \tag{2}$$

whence $U_j(s) = (g(s))^{-j}\, u_j(s)$ and

$$\Pi_k(m,s) = (g(s))^k\, u_{m-k}(s)/u_m(s). \tag{3}$$

Finally define $\{v_j(s): j=0,1,\ldots\}$ by $v_0(s) \equiv 1$ and $v_j(s) = u_j(s) - u_{j-1}(s)$ $(j=1,2,\ldots)$ and hence

$$\sum_{j=0}^{\infty} v_j(s)t^j = \frac{p_0 s(1-t)}{g(s)(\bar{f}(t)-t)} \tag{4}$$

and $v_j(s) \geq 0$ $(0 < s \leq 1)$; see, for example, Yang (1973, p.448).

It follows from (3) that

$$P(M \leq x \mid S_0=k) = u_{[x]-k}/u_{[x]} \tag{5}$$

and Green (1976) has shown that

$$E(s^N \mid M > j) = g(s)v_j(s)u_j/v_j u_j(s) \tag{6}$$

and hence that knowledge of the asymptotic behavior of the u's and v's yields limit theorems for M and N. Here $v_j = v_j(1)$ and $u_j = u_j(1)$.

It is pointed out in Pakes (1977) that

$$\sum_{n,j\geq 1} p_{1j}^{(n)} s^n t^j = \frac{tg(s)-t^2}{sf(t)-t} \tag{7}$$

where $p_{ij}^{(n)} = P(S_n=j \mid S_0=i)$, and hence

$$\sum_{n,j\geq 1} p_{1j}^{(n)} s^n (g(s))^j t^j = g(s)\frac{t(1-t)}{\bar{f}(t)-t}.$$

Thus we obtain the representation

$$v_j(s) = p_0 s(g(s))^{j-1} \sum_{n\geq 1} p_{1,j+1}^{(n)} s^n.$$

Assume now that $\alpha = 1$ and $b = f''(1-) < \infty$. We see that

$v_j = p_0 G_{1,j+1}$ where $G_{1j} = \sum_{n\geq 1} p_{1j}^{(n)}$ is a Green's function.

Lemma 1 of Pakes (1977) then yields

$$v_j \to 2p_0/b\ , \quad u_j \sim 2p_0 j/b. \tag{8}$$

The first relation is a direct consequence of a discrete renewal theorem; we shall enlarge on this below.

Let now $s = s_j = \exp(-\theta b/2j^2)$ $(j=1,2,\ldots)$. It is easily seen that $\{g(s_j)\}^j \to \exp(-\theta^{1/2})$. We now make use of the local limit theorem for $p_{1j}^{(n)}$ derived in Pakes (1977):

$$\sup_{j\geq 0} |bnp_{1j}^{(n)} - (2/\pi b)^{1/2} jn^{-1/2}\exp(-j^2/2bn)| \to 0 \quad (n\to\infty). \tag{9}$$

Observe that $p_j(n) = p_{1j}^{(n)}/G_{1j}$ defines the distribution of a random variable (r.v.), $A(j)$ say, via $p_j(n) = P(A(j) = n)$.
Letting $n = 2x_j j^2/b$ in (9), where $x_j \to x > 0$, we obtain from (8) and (9)

$$(2j^2/b)p_j(n) \to \sigma(x) \equiv (2\pi^{1/2})^{-1}\, x^{-3/2}\, \exp(-1/4x)$$

which is the density of the stable law whose Laplace-Stieltjes transform (L.S.T.) is $\exp(-\theta^{1/2})$. It follows (Billingsley (1968) Theorem 7.8) that

$$P(bA(j)/sj^2 \leq x) \to \int_0^x \sigma(y)dy$$

and finally that

$$v_j(s_j) \to (2p_0/b)\exp(-2\theta^{1/2}) \qquad (j\to\infty). \tag{10}$$

Note that the convergence here is uniform for $0 \leq \theta \leq A < \infty$.

Let $\psi_j(\theta) = v_j(s_j)$ and $\psi(\theta)$ represent the right hand side of (10). Then

$$j^{-1}u_j(s_j) = j^{-1} + j^{-1}\sum_{k=1}^{j}[\psi_k(\theta k^2/j^2) - \psi(\theta k^2/j^2)]$$

$$+ j^{-1}\sum_{k=1}^{j}\psi(\theta k^2/j^2). \tag{11}$$

As $j\to\infty$ the last term on the right

$$\to (2p_0/b)\int_0^1 \exp(-2x\theta^{1/2})\,dx = (p_0/b)\theta^{-1/2}(1-\exp(-2\theta^{1/2})).$$

Let $0 > \varepsilon > 1$ be given. Then the second term on the right of expression (11)

$$\leq 2\varepsilon + j^{-1}\sum_{\varepsilon j\leq k\leq j}|\psi_k(\theta k^2/j^2) - \psi(\theta k^2/j^2)|$$

and j can be chosen so large that the summands $\leq \varepsilon$.

Thus we find that

$$j^{-1}u_j(s_j) \to (p_0/b)\theta^{-1/2}(1 - \exp(-2\theta^{1/2})).$$

Combining this with (8) and (10) in (6) yields

<u>Theorem 1</u>. <u>Let</u> $\alpha = 1$ <u>and</u> $b = f''(1-) < \infty$. <u>Then</u>

$$P(bN/2j^2 \leq x|M > j) \to F(x) \qquad (j\to\infty)$$

<u>where</u> F <u>has L.S.T.</u> $(\exp(-\theta^{1/2}))\,\theta^{1/2}\text{cosech}\,\theta^{1/2}$.

Assume now that

$$f(t) = t + (1-t)^{\delta}\, L((1-t)^{-1}) \tag{12}$$

where $1 < \delta \leq 2$ and $L(\cdot)$ is slowly varying (s.v.) at infinity.

We then have

$$\sum_{j=0}^{\infty} u_j t^j = p_0/(1-t)^{\delta}\, L((1-t)^{-1})$$

and since $\{u_j\}$ is a non-decreasing sequence, a Tauberian theorem for power series yields

$$u_j \sim p_0 j^{\delta-1}/\Gamma(\delta)L(j) \qquad (j\to\infty).$$

The following result is now an immediate consequence of (5).

Theorem 2. Suppose that (12) holds. Then

$$P(M \le kx | S_0=k) \to [(1 - x^{-1})^+]^{\delta-1} \qquad (k\to\infty).$$

Equation (4) yields $\sum v_j t^j = p_0/(1 - h(t))$ where $h(t) = (1 - f(t))/(1-t)$ is a p.g.f. The discrete renewal theorem (Erickson (1970), eq.(2.3)) shows that if (12) holds and if $3/2 < \delta \le 1$ then $jv_j/u_j \to \delta-1$. This also holds when $1 < \delta \le 3/2$ because the weights of the distribution defined by h are monotonic and corollary 3A of Williamson (1968) then applies. We now have the following generalization of results of Green (1976) and Lindvall (1976) on the tail behavior of M.

Theorem 3. Suppose (12) holds. Then

$$\lim_{j\to\infty} jP(M > j | S_0=k) = (\delta-1)k.$$

3. THE CASE OF NEGATIVE DRIFT

In this section we assume $\alpha < 1$. The p.g.f. $\overline{f}(t)$ has mean $1 - g(s)/sg'(s) < 1$ $(0 < s \le 1)$ and hence (2) and the Tauberian theorem for power series implies that

$$u_j(s) \uparrow p_0 g'(s)(s/g(s))^2 \qquad (0 < s \le 1;\ j\to\infty). \qquad (13)$$

Furthermore the limit function is continuous in $[1 - \varepsilon, 1]$ $(0 \le \varepsilon < 1)$, whence by Dini's theorem the convergence at (13) is uniform with respect to $s \in [1-\varepsilon, 1]$. In particular we have

<u>Lemma 1</u>. <u>If</u> $\alpha < 1$ <u>and</u> $s_j \le 1$, $s_j \to 1$ $(j \to \infty)$ <u>then</u>

$$u_j(s_j) \to p_0/(1-\alpha).$$

Now assume that there exists a solution, D, exceeding unity of the equation $f(t) = t$ and that the p.g.f. $f_D(t) = f(Dt)/D$ can be expanded as

$$f_D(t) = 1 - \delta^{-1}(1-t)^{\delta} L\big((1-t)^{-1}\big) \qquad (0 < \delta < 1)$$

or

$$f_D(t) = 1 - a(1-t) + \delta^{-1}(1-t)^{\delta} L\big((1-t)^{-1}\big) \quad (1 < \delta \le 2)$$

where in the latter case $1 < a = f'(D-) < \infty$ and $L(\cdot)$ is S.V. at infinity. In either case the least positive solution, q, of $f_D(t) = t$ is $q = D^{-1}$. Let $g_D(s) = qg(s)$; we have $g_D(s) = sf_D\big(g_D(s)\big)$. Substitute tD for t in (7) to obtain

$$\sum_{n,j\ge 1} p_{1j}^{(n)} s^n D^t t^j = Dt \frac{g_D(s)-t}{sf_D(t)-t} = D \sum_{n,j\ge 1} q_{1j}^{(n)} s^n t^j \qquad (14)$$

where $[q_{ij}^{(n)}]$ is the n-step transition matrix of an absorbing origin left-continuous random walk, $\{Y_n\}$, with increment p.g.f. $f_D(t)/t$ and hence has positive drift. Thus we obtain

$$v_j(s) = p_0 s\big(g(s)\big)^{j-1} D^{-j} \sum_{n\ge 1} q_{1,j+1}^{(n)} s^n , \qquad (15)$$

and setting $Q_{1j} = \sum_{n\ge 1} q_{1j}^{(n)}$ we have

$$D^j v_j = p_0 Q_{1,j+1}. \qquad (16)$$

Equation (14) yields $\sum_{j\geq 1} Q_{1j}t^j = t(q-t)/(f_D(t)-t) = t/(1-d(t))$ where $d(t) = (f_D(t)-q)/(t-q)$ is a p.g.f. Application of the discrete renewal theorem yields

<u>Lemma 2</u>. <u>If</u> $1/2 < \delta < 1$ <u>then</u>

$$j^{1-\delta} L(j)\, Q_{1j} \to (1-q)\delta/\Gamma(\delta)$$

<u>and if</u> $\delta > 1$ <u>then</u>

$$Q_{1j} \to (1-q)/(a-1).$$

The asymptotic behavior of $\{v_j\}$ now follows from (16). It is shown in Pakes (1973) that

$$P(Y_n \leq xB(n) + \Delta n \mid Y_0=1) \to q + (1-q)\, H(x) \qquad (n\to\infty)$$

where $\Delta = 0$ if $\delta < 1$, $\Delta = a-1$ if $\delta > 1$ and $H(\cdot)$ is the D.F. of the stable law whose characteristic function is

$$\phi(\theta) = \exp[\delta^{-1}(\cos(\pi\delta/2))|\theta|^{\delta}(1-i\theta|\theta|^{-1}\tan(\pi\delta/2))],$$

$B(\cdot)$ is the inverse function of $x^{\delta}/L(x)$ and has the form $B(n) = n^{1/\delta}M(n)$ where $M(\cdot)$ is S.V. at infinity. Using the representation for $\sum q_{1j}^{(n)}t^j$ given in §6 of Pakes (1973) and mimicking the proof of the local limit theorem for lattice distributions as presented, for example, by Ibragimov and Linnik (1970), it can be readily checked that the following holds:

$$\sup_j |B(n)q_{1j}^{(n)} - (1-q)h((j-\Delta n)/B(n))| \to 0 \qquad (n\to\infty) \qquad (17)$$

where $h(x) = H'(x)$. We must now distinguish a number of cases.

<u>(i). $1/2 < \delta < 1$.</u> Let $n = x_j j^{\delta}/L(j)$ where $x_j \to x$. Clearly $B(n) \sim jx^{1/\delta}$, and hence if $\{T_j\}$ is a sequence of r.v.'s such that

$P(T_j=n) = q_{1j}^{(n)}/Q_{1j}$, we obtain

$$\left(j^{\delta}/L(j)\right)\ P\left(T_j=x_j j^{\delta}/L(j)\right) \to p(x) \tag{18}$$

where $p(x) = \delta^{-1}\Gamma(\delta)x^{-1/\delta}\ h(x^{-1/\delta})$ is a density function. This may be seen by observing that $\int_0^\infty x^{-1/\delta}h(x^{-1/\delta})dx = \delta M_h(1-\delta)$ where $M_h(\theta) = \int_0^\infty x^{\theta-1}h(x)dx$ is the Mellin transform of $h(\cdot)$. Now use the formula (Kawata (1972), p. 272)

$$M_h(\theta) = \left(\Gamma(1-\theta)\right)^{-1}\int_0^\infty x^{-\theta}\lambda_h(x)dx\ ,$$

where $\lambda_h(\theta) = \exp(-\delta^{-1}\theta^{\delta})$ is the L.S.T. of $H(\cdot)$, to obtain $M_h(1-\delta) = \left(\Gamma(\delta)\right)^{-1}$. Indeed it is easy to show that $M_p(\theta) = \Gamma(\delta)\Gamma(\theta)/\delta^{1-\theta}\Gamma(\delta\theta)$. Lemma 2 and (18) now yield

$$Q_{1j}^{-1}\sum_{n=1}^{\infty} q_{1j}^{(n)}\ \exp\left(-\theta n j^{-\delta}L(j)\right) \to \xi(\theta)$$

where $\xi(\theta) = \int_0^\infty e^{-\theta x}p(x)dx$.

Under our present assumptions we have $g''(1-) < \infty$ and $g'(1-) = A = (1-\alpha)^{-1}$. It is readily checked that the condition $1/2 < \delta < 1$ implies

$$[\exp\left(A\theta j^{1-\delta}L(j)\right)][g\left(\exp\left(-\theta j^{-\delta}L(j)\right)\right)]^j \to 1.$$

Using (15) and (16) we can rewrite (6) as

$$E(s^N|M>j) = \left(g(s)\right)^j\ \left(Q_{1,j+1}^{-1}\sum_{n=1}^{\infty} q_{1,j+1}^{(n)}\ s^n\right)\left(u_j/u_j(s)\right)$$

and the subsequent result readily follows.

<u>Theorem 4</u>. <u>If</u> $1/2 < \theta < 1$ <u>and</u> $\nu(j) = j^{\delta}/L(j)$ <u>then</u>

$$P(N - Aj \le x\,\nu(j)|M > j) \to \int_0^x p(y)dy \qquad (j\to\infty;\ x>0).$$

<u>(ii). $1 < \delta \le 2$.</u> Now let $n = j/\Delta + x_j B(j)$ in (17), whence

$$B(j)P\left(T_j = j/\Delta + x_j B(j)\right) \to \Delta^{1+1/\delta}\ h(-x\Delta^{1+1/\delta})$$

and finally

$$[\exp(\theta j/\Delta B(j))]\ Q_{1j}^{-1} \sum_{n\ge 1} q_{1j}^{(n)} \exp[-\theta n/B(j)] \to \zeta(\theta)$$

where $\zeta(\theta) = \exp[\delta^{-1}\theta^{\delta}\Delta^{-1-\delta}]$ is the moment generating function $\int_{-\infty}^{\infty} (\exp\,\theta x)\ \Delta^{1+1/\delta}\ h(-x\Delta^{1+1/\delta})dx$. A special case is $f''(D-) < \infty$ and $b = Df''(D-) + a - a^2$ which implies that $\delta = 2$, and $B(j) = (bj)^{1/2}$.

We must now consider two cases. First assume that either $1 < \delta < 2$ or if $\delta = 2$ then $L(x) \to \infty$. Then in a similar manner to case (i)

$$[\exp(A\theta j/B(j))][g\left(\exp(-\theta/B(j))\right)]^j \to 1$$

which follows on observing that $M(x) \to \infty$ when $\delta = 2$. The following result is now apparent.

<u>Theorem 5.</u> <u>If</u> $1 < \delta \le 2$ <u>and</u> $f''(D-) = \infty$ <u>then</u>

$$P(N - Aj \le xB(j)|M > j) \to 1 - H(-x\Delta^{1+1/\delta}) \quad (-\infty<x<\infty).$$

For our final case assume that $f''(D-) < \infty$. Letting $B = \left(f''(1-) + \alpha - \alpha^2\right)/(1 - \alpha)^3$ we have

$$[\exp(A\theta j^{1/2})][g\left(\exp(-\theta j^{-1/2})\right)]^j \to \exp(B\theta^2/2).$$

Now let $\mu = A + \Delta^{-1}$ and $\sigma^2 = B + b\Delta^{-3}$. The following result readily follows.

Theorem 6. *If* $f''(D-) < \infty$ *then*

$$P(N - \mu j \le x\sigma j^{1/2} | M > j) \to \Phi(x), \qquad (-\infty<x<\infty)$$

the standard normal distribution function.

This is Green's Theorem 3 but he assumed that $f(s)$ is finite in $|s| < D + \varepsilon$ for some $\varepsilon > 0$. Lemma 1 and (5) yield the following analogue of Theorem 2.

Theorem 7. *Assuming only that* $\alpha < 1$,

$$\lim_{k\to\infty} P(M \le kx | S_0=k) = 0 \ (0 \le x < 1), = 1 \quad (x > 1).$$

Finally Lemmas 1 and 2 yield the following partial refinement of Green's Theorem 1.

Theorem 8. *If* $f'(D-) < \infty$ *then*

$$\lim_{j\to\infty} D^j P(M > j | S_0=k) = (D^k-1)(1-\alpha)/(a-1).$$

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

In a recent paper P.J.Green obtained some conditional limit theorems for the absorption time of left-continuous random walk. His methods required certain distributions to have exponentially decreasing tails. Here a different approach is taken to produce Green's results under minimal conditions. Limit theorems are given for the maximum as the initial position of the random walk tends to infinity.

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